Quantum Computing for Beginners

Paul Pham with illustrations by Stephen Rice

October 5, 2012

0.1 Instructions

Reading these entire notes and doing the problem set is meant to take at most two hours of your time before class. You don't need to read the Historical Notes, they are provided if you are curious for more details. Keep careful track of how long you spend on this assignment. If it takes you more than two hours, feel free to stop and come to class prepared to ask questions and finish the assignment with your classmates.

Chapter 1

In which Elina meets Ehrenfest, Planck, and a quantum bit

1.1 Göttingen, Germany: September 1930

Elina steps hesitantly into the cafe. It is the middle of an autumn afternoon, and the rays of the sun are already slanting through the door behind her to light up the tables inside. It complements the Bauhaus minimalism that is currently all the rage in Germany, but the cafe interior looks too bare and unadorned compared to the teahouses in Odessa. She scans the patrons for the man she is looking for and heads toward the first one that she sees scribbling in a notebook. He is a squat but gentle-looking man with dark hair and a thick, straight line of a mustache across his upper lip. He looks to be about 50 years old. He is muttering to himself under his breath as he crosses out some symbols. When she is closer she sees coffee-stained pages of equations in a hasty longhand.

"P-pardon, are you Herr Professor Planck?" she asks softly, making only occasional eye contact, still a little embarrassed by her Russian accent.

The man looks up in surprise at her question, and then becomes even more surprised when he sees her. With a gentle smile, he replies, "Well, fraulein, what if I were?"

"Sorry, I see the equations, so I thought..." she trails off. Thinking she has the wrong man, she turns to go.

He smile widens. "In Göttingen, *fraulein*, everyone and his dog does mathematics of some kind. But still, you would be very lucky to walk into a cafe and find Max Planck sitting waiting for you." Then a little sadly he says, "No, I am not the Herr Professor Doktor, although sometimes I wish I were. I am called Paul, Paul Ehrenfest." He searches her face for recognition, but she stares back blankly. “Oh, but you won’t have heard of me. No matter. I can take you to Planck. I’ve just arrived on the train from Leiden to visit him myself. But first, may I know *your* name?”

“Oh, of course! Thank you,” Elina is both embarrassed and relieved. “Please, Herr Ehrenfest, I am called Elina Gamow.”

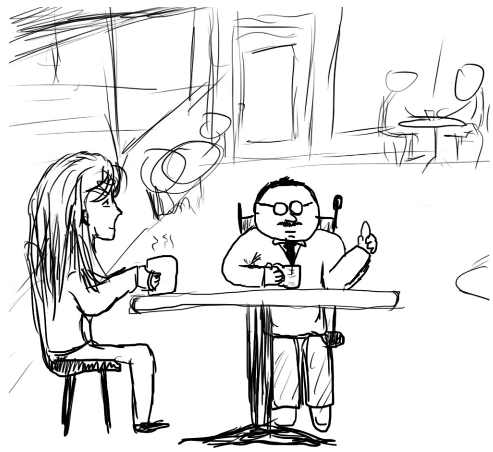

Figure 1.1: Elina and Ehrenfest at a Bauhaus-style cafe near the Göttingen train station.

“Gamow... Gamow... ” Ehrenfest tries to place the name. “Are you related to George Gamow, the great Russian physicist?”

“You mean Uncle Georgiy? I did not know he was so great,” she says mostly to herself, then blurts out, “I mean, he is a great *uncle*.”

“Ha, and you are a good niece to say so.” Ehrenfest’s manner has become more open now that he has connected her to the world of physics. “Have you arrived to study the new quantum theory?”

“Pardon, sir, no, I mean, I don’t know. I am here to work for Herr Professor Planck.” She is quick though, and has remembered the unfamiliar word, even though she doesn’t know what it means. “What is a quan-tum? Will it help me with the keypunch machines?”

“Hmm, possibly, um, er, I think,” Ehrenfest hems and haws and he can see the skepticism on Elina’s face. “Actually, yes! My dear friend Albert Einstein—”

“Einstein!” she exclaims, finally recognizing another physicist besides her uncle. “You are friends with Einstein?”

“Yes, of course, you would know him. Einstein has just received a splendid position at Princeton, and he tells me the loud Hungarian in the office next door has done some thinking about quantum mechanics and how to build a machine using its special properties. I will write to him at once and ask him to explain some of his ideas, for I have been meaning to learn about them, too."

"If it's not too much trouble," Elina says, and Ehrenfest can tell she is not quite won over yet.

"Anyway," he continues, "you seem too bright merely to be a keypunch operator. Your uncle was probably hoping that would learn a little bit about his research. Surely that's what he had in mind, after all, Niels Bohr is coming to town next week to give a series of lectures on quantum physics. Anyone who is anyone in the physics world will be here. If you want to learn about quanta, Göttingen is the best place to be right now.

"Well let's save that for later. We'll go to Planck soon, but do you mind if I finish this calculation first? It is just a theorem I have been working on. I will order you a coffee to drink while you wait."

Elina accepts his offer graciously, and later that day she finds herself in the office of Max Planck. She is sitting in front of a large wooden desk, feeling a little out of place. She looks down at the letter of introduction she is clutching in her right hand so tightly that she has forgotten she is holding it. Now she thrusts it forward across the desktop to Planck.

Figure 1.2: Elina meets Max Planck.

He is a small, wizened old man, balding, with very round eyeglasses. At this point he is 71 years old, but still with a bright, attentive gaze. He reads her letter of introduction meticulously before handing it back to her. "Yes, of course, I remember Gamow writing to me several weeks ago. He was a very bright student when he was here, and I hear he is doing quite well in Copenhagen. I'm glad that you arrived safely. We are quite behind on tabulating the data from our experiments. Did you have any trouble finding the physics institute?"

"No, Herr Professor, I was lucky enough to meet Herr Ehrenfest in the cafe near the train station—" Elina begins.

"Ehrenfest? Has he arrived as well?" Planck interrupts. "It seems everyone is arriving for these lectures by the Dane, Niels Bohr. Ah well, never mind, you won't be drawn into this quantum heresy in your own work. You'll be using good, simple, old-fashioned classical calculating machines. Ah, sometimes I regret ever starting that whole quantum business."

1.2 Technical Coda: What is a Quantum Bit?

Figure 1.3: Einstein's loud Hungarian coworker at Princeton.

September 27, 1930

Dear Elina,

You've been mentioned to me by Einstein as someone who wants to understand how we can use quantum physics to build a (potentially useful) machine. I also share your interest! The physicists have probed at Nature to discover that she obeys this strange quantum model. I have been thinking about it from the point of view of mathematics and the computation of functions. I don't know if it will ever be useful, but I am happy to share what I have discovered with you.

This letter is quite long and may contain many words and ideas that are unfamiliar to you at first. Please be patient, and your effort will be rewarded with a beautiful understanding. I've included some practice problems at the end of the letter which you may find useful to test yourself. So without further ado, I will present to you some ideas that I have been throwing around in case any of them stick. Please keep in mind that these are very tentative, so be kind in your criticism! Oh, and in this letter, I will use the terms "quantum mechanics" and "quantum physics" interchangeably, but mechanics is just the area of physics traditionally concerned with motion, position, and matter. You will see later why this distinction breaks down in quantum physics.

In the keypunch machines which you are operating in your job, each position either has a hole punched into a card, or is just unpunched paper. It is definitely, with 100% probability, in one of two configurations or states. John Tukey, a bright fellow here in Princeton, has taken to calling this unit of information a *bit*, short for binary digit, since these states can represent the values of 1 and 0. I've become quite fond of the term as well. You can even think of it as a lightswitch which is on or off. I describe a bit as *classical* because it is based on the classical physics which we have all studied in school from the time of Isaac Newton.

In quantum systems, physicists tell me we can consider something similar to a bit, but with some extra features, as it were. I won't go into details of how the physical system looks, but I will give you an abstract model that you can compare to the lightswitch.

Imagine a sphere, like the globe of the earth, where the north pole is the state 0 and the south pole is the state 1. Instead of a bit, which can only take on the two states at the poles, a *quantum bit* can take on any position on the surface of the sphere. You can think of this quantum bit, or qubit, as a mixture of a zero state and a one state. This is sometimes called a Bloch sphere, especially by friends of the physicist Felix Bloch! Now don't take this picture too seriously, it has some shortcomings and only applies to a single qubit by itself, but it will be a useful scaffold for learning.

I'll use a notation invented by Paul Dirac at Cambridge for writing quantum states called *kets*. A zero state is written as $|0\rangle$, which we read as "ket zero" or simply "zero" when it is understood that we are only talking about quantum states. Likewise, a one state is written as $|1\rangle$. Kets distinguish qubits from classical bits.

Mathematically, a qubit is also a *vector* in a two-dimensional vector space where the standard basis is $|0\rangle$ and $|1\rangle$. I'll use kets and vectors interchangeably to refer to quantum states. In order to make it easy to do calculations (for example, in an automated way by encoding numbers on punched cards), we write a qubit as a two-element *column vector* as follows in Equation 1.1, with the first element being the "zeroness" of a qubit and the second element being the "oneness" of a qubit. Using vectors and numbers is often called an *algebraic* way of thinking about a quantum bit, whereas the Bloch sphere is called a *geometric* way of thinking about the same concept. The two views are

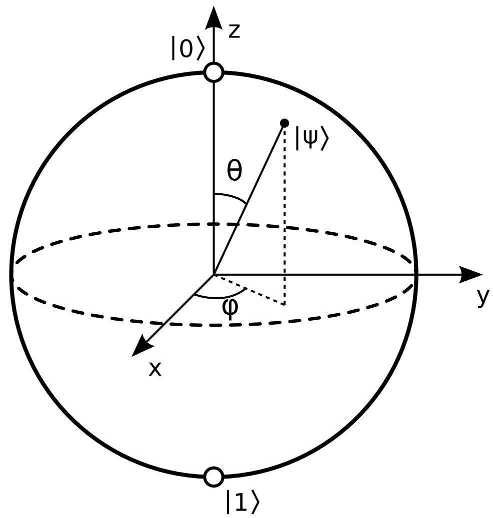

Figure 1.4: The state of a quantum bit can be represented by a point on the surface of the Bloch sphere.

equivalent up (to a point).

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1.1}$$

The state $|0\rangle$ is 100% zero and not one at all. The state $|1\rangle$ is 100% one and not zero at all. These are analogous to the classical bits 0 and 1. A generic qubit is usually denoted with a Greek letter psi ($|\psi\rangle$) and is some combination of $|0\rangle$ and $|1\rangle$, with those components written as α and β.

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{1.2}$$

We can transform a qubit by performing an operation, or a logic "gate" in the language of electrical circuits. This gate is represented by a two-by-two square matrix.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1.3}$$

We can model the act of performing a gate on a qubit with multiplying a matrix by a column vector.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} \tag{1.4}$$

The matrix that corresponds to the "identity" gate, or doing nothing and is simply the identity matrix.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.5}$$

You can verify that it leaves a qubit unchanged.

To complete my assertion that the state of a quantum bit is a point on the surface of the Bloch sphere, we need to impose two more conditions on our mathematical model.

The first condition is that α and β are *complex* numbers, which means they have a real and an imaginary part. Remember that the imaginary unit i is the square root of -1!

$$\alpha = a + bi \tag{1.6}$$

$$\beta = c + di \tag{1.7}$$

The *magnitude* of a complex number is simply the square root of the sum of the components squared. We write the magnitude of a complex number α as $|\alpha|$, and note that it is a real number.

$$|\alpha| = \sqrt{a^2 + b^2} \tag{1.8}$$

$$|\beta| = \sqrt{c^2 + d^2} \tag{1.9}$$

If α and β were not complex, and instead just real numbers, then instead of a quantum state corresponding to the surface of a sphere, it would be restricted to the edge of a circle.

The second condition is that the sum of the magnitudes squared of α and β must be one. It could be any constant number as far as you know now, but take my word for it that it should be one. Therefore, a quantum bit which is a mixture of one-fourth in the state $|0\rangle$ and three-fourths in the state $|1\rangle$ might have the following amplitudes.

$$\begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \tag{1.10}$$

Well, this letter has quite gotten away from me. I did not mean to say so much in one go, but there it is. Try your hand at the problem to follow, and please feel free to ask many questions. I look forward to your response, and hope that you are as interested in exploring the possibilities of this new quantum bit as I am.

Best, Johnny

1.3 Problem Set

1. Contemplate the beauty of this equation, which is called Euler's jewel by Richard Feynman. It relates five of the most important numbers in all of mathematics, the base of the natural logarithm *e*, the imaginary unit *i*, the ratio between a circle's circumference and its diameter π, and the two possible values of a bit: 0 and 1.

$$e^{i\pi} + 1 = 0 \tag{1.11}$$

This is an algebraic description of the relationship between all these numbers. An elegant geometric description of this relationship is given by this diagram of the *unit circle*, which is a two-dimensional circle centered on the origin $(0, 0)$ where the coordinates represent real numbers on the x-axis, imaginary numbers on the y-axis, and we restrict our attention to the circle of all points that are of *unit distance* from the origin (that is, they have a distance of 1 from the origin).

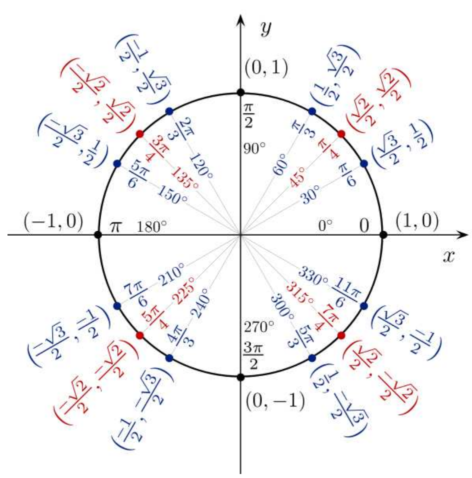

Figure 1.5: The unit circle is a geometric picture relating complex phases to angles, the Cartesian grid, trigonometry, and the infinite series that calculates *e*.

Every complex number on the unit circle can be written as $e^{i\phi}$, where ϕ is some angle between 0 and 2π radians. A radian, you'll recall, is the angle where the arc on a unit circle has the length 1. A full rotation around a circle is 2π radians.

What are the real/imaginary coordinates in the form (x, y) for the four numbers 1, -1, i, and $-i$ on the unit circle? What are the angles ϕ to express the same four numbers as $e^{i\phi}$?

2. We know that $|0\rangle$ and $|1\rangle$ are the north and south pole of the Bloch sphere respectively, with the column vectors in Equation 1.1. The "equator" of the Bloch sphere is just the unit circle from the previous question! There are four column vectors that correspond to four points on this equator, all equally spaced, which we will write as follows:

$$|\psi_1\rangle = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \quad |\psi_2\rangle = \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \quad |\psi_3\rangle = \begin{bmatrix} \alpha_3 \\ \beta_3 \end{bmatrix} \quad |\psi_4\rangle = \begin{bmatrix} \alpha_4 \\ \beta_4 \end{bmatrix} \tag{1.12}$$

These four points are, in a sense, equal superpositions of $|0\rangle$ and $|1\rangle$ but may have different "phases" relative to each other, which just means they occupy different points on the unit circle.

What are the values of α_i and β_i above, for $i \in \{1, 2, 3, 4\}$? Here are some clues to get a unique solution.

α_1, β_1, α_2, and β_2 are completely real. α_3 and α_4 are real but β_3 and β_4 are imaginary. β_1 plus β_2 is zero. β_3 plus β_4 is zero.

3. What is the matrix for the gate (call it M_1) which "flips" a qubit as follows?

$$M_1 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix} \tag{1.13}$$

What is the matrix for the gate (call it M_2) which adds a negative phase to one component of a qubit as follows?

$$M_2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix} \tag{1.14}$$

4. A global phase $e^{i\phi}$, that is, any complex number, can be factored out of any quantum state, which is just a case of dividing out a common factor from the elements of a column vector or a matrix. For example, the following two states are the same:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} i \\ -i \end{bmatrix} \tag{1.15}$$

What is the effect of the following gate on the states $|0\rangle$ and $|1\rangle$, and on the states $|\psi_1\rangle$ and $|\psi_2\rangle$ from Equation 1.12?

$$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{1.16}$$

That is, what is the resulting vector when the gate is applied to each of those four states, and where are these resulting states on the Bloch sphere? How would you describe the action of this gate geometrically, as a rotation about some axis?

5. Who is the historical figure writing the letter to Elina? Describe one of his contributions to computer science, perhaps one that you have learned about in a previous class.

1.4 Credits and Historical Notes

Much of the historical research was taken from Wikipedia articles and confirmed from folklore and anecdotes told to the author throughout his education. Wikipedia also provided the SVG sources for the images of the Bloch sphere and the unit circle.

Elina Gamow is a completely fictional character, although her uncle George Gamow is real and is famous for discovering the decay of alpha particles (the nucleus of the helium atom) by quantum tunneling.[1] However, her experiences and meetings with real things, places, people is meant to be typical and realistic for the time period. Many young, unmarried women were hired to operate punched card machinery, which was the leading information processing technology in the 1930s, although I made up the fact that they needed keypunch operators for experimental data at Göttingen, which was not known for its experimental physics. Ironically, these women performed many of the tasks we now consider the job of a computer programmer, including translating human problems into a machine-readable format, debugging errors, and interpreting results. It is unfortunate that today, computer programming and software engineering is considered a stereotypically male profession.

Real things, places, and people: Göttingen is a real city in Germany, and was, in fact, famous as a center of mathematics and science during the early half of the 20th century. Einstein called it the "Eldorado of erudition." Many physicists who helped develop the quantum theory studied here as undergraduates or Ph.D. students, including George Gamow and Werner Heisenberg, and Max Planck himself was an actual faculty member there in 1930. Max Planck coined the term "quantum" when he famously proposed that heat is absorbed and emitted in discrete energy packets, thus resolving the long-standing open problem of *blackbody radiation* for which he won the Nobel Prize in 1918. He would already have been a venerated and established scientific figure at the time of our story. He was famously conservative in both his social and scientific views, and for the rest of his life he (along with the much younger Einstein) always resisted the full implications of the quantum theory which he helped kickstart.

Paul Ehrenfest was of the same, older generation of quantum physicists, somewhere between the ages of Planck and Einstein. He was a close friend to both Einstein and Bohr, and most of our knowledge of the personal lives of all three men come from their extensive letters to each other. Although Ehrenfest is not as famous as the other physicists in this story, he made several significant scientific contributions as well as mentoring many younger physicists, most famously the Italian physicist Enrico Fermi who later worked on the Manhattan Project. Einstein often commented upon Ehrenfest's enormous empathy for others and his skills as a teacher and listener: "He was not merely the best teacher in our profession whom I have ever known; he was

[1] Gamow was very interested in teaching later in life, wrote many popular science books, and had a great sense of humor. The author hopes he would have approved of this class and his imaginary niece.

also passionately preoccupied with the development and destiny of men, especially his students. To understand others, to gain their friendship and trust, to aid anyone embroiled in outer or inner struggles, to encourage youthful talent all this was his real element, almost more than his immersion in scientific problems." He was in fact a professor at Leiden for most of his career.

Niels Bohr did arrive in Göttingen to give a famous series of lectures on the then new quantum physics, for which Paul Ehrenfest and many other physicists traveled across Europe to hear. However, that was in 1922, and not in 1930 as in the story.

In reality, the term "qubit" was coined by Ben Schumacher. The representation of a quantum bit with the Bloch sphere corresponds to the homomorphism between the groups $SU(2)$ and $SO(3)$ and requires some more mathematical sophistication to understand completely. For more details, see problem 2 in the first homework from Aram's class. The exposition presented here is just meant to give you a flavor of what makes a qubit different from a classical bit.

John Tukey was a prominent statistician who did coin the term "bit," but in 1948 while working on early digital computers. He was an undergraduate at Brown during the time of this story. He is also famous for developing the fast Fourier transform (FFT) algorithm which is a key concept in modern computer science, digital signal processing, and computer engineering.

Other historical notes will have to wait for the next chapter so as not to ruin too many surprises.