Quantum Computing for Beginners

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0.1 Instructions

Reading these entire notes and doing the problem set is meant to take at most two hours of your time before class. You don't need to read the Historical Notes, they are provided if you are curious for more details. Keep careful track of how long you spend on this assignment. If it takes you more than two hours, feel free to stop and come to class prepared to ask questions and finish the assignment with your classmates.

Chapter 4

In which we meet Pauli

4.1 Göttingen Hauptbahnhof (Central Train Station)

Dirac has taken a seat next to Ehrenfest and Elina at the cafe, tired from his soliloquy on vector spaces, inner products, and bra-ket notation. He

"I normally don't speak very much," he assures his companions.[1] "But this letter from Von Neumann that you showed me mentions this interesting tensor product structure. How can you use it to study larger quantum states?"

"Oh, I just finish solving this problem," Elina says eagerly, showing him the pages in her notebook where she has been working out some problems with two-qubit states. Dirac reads her letter in silence, when a large, rotund man with a broad, bald forehead walks into the cafe.

"Dirac, there you are!" the fat man shouts loudly and walks over, his coat draped over his arm and clutching a newspaper. "I almost didn't recognize you, sitting with other people, and an attractive young woman at that. The train from Heilbronn was very late, I had to go five hours without a drink. Barbaric, it was!" He turns to the waiter and immediately shouts, "You there, boy, bring me a glass of red wine! And make it quick."

Dirac looks up from the equations in annoyance until he recognizes the face of his friend. "Pauli, you've arrived finally. I was hoping to run into you." He turns to his table companions. "May I present to you Wolfgang Pauli, who is a professor of physics in Zurich. This is Paul Ehrenfest, from Leiden."

The thick-haired Ehrenfest rises slightly with a bow. "Herr Pauli, I am a great admirer of your encyclopedia article on Einstein's general relativity."

"Ehrenfest, eh?" Pauli pulls out a cigar from his inside pocket and lights it with a flourish. "I didn't like *your* encyclopedia article at all." He notices the startled look on the other man's face and begins to laugh. "But *you* seem all right." Everyone at the table freezes, afraid of an awkward situation, but then Ehrenfest laughs.

[1] At Cambridge, his colleagues define the dirac as the unit of one word per hour.

"Herr Pauli, with me, regarding you, I feel it is just the opposite."

They both laugh, and Dirac relaxes. The waiter arrives just then with a wineglass on a tray, and Pauli grabs it with his free hand and downs it in one gulp. "Another, please." The waiter's eyes widen and he takes the empty glass away again. "And the fraulein?"

"This is Elina Gamow, the niece of George Gamow," Dirac says formally.

"It pleases me to meet you, Herr Professor—" Elina begins.

"Your German is terrible!" Pauli interrupts sternly, but then he smiles again. "But I forgive you, for the sake of your uncle, and because you have only these two to teach you."

Dirac sees the girl is surprised at the insult at first. "Don't worry, Miss Gamow, Pauli insults everyone. You should only be sad if he ignores you."

"That's right, fraulein, Dirac knows because he is very good at being ignored." Pauli chuckles wryly at his friend. "But now we are only missing Heisenberg from our happy trio. Surely he wouldn't miss hearing the Dane[2] He's the one who needs this education the most."

"He is supposed to be on the train from Leipzig, to arrive this afternoon at 4:22," Dirac says precisely.

"Well, it's too bad he is not here now. I wanted to discuss politics with him." The fat man throws down the newspaper on the table, upsetting Dirac's papers, and sits down. The second glass of red wine arrives. Pauli sips it this time, and takes a puff from his lit cigar. "Election season has just passed in Germany, I don't suppose you've been keeping up at all?"

Ehrenfest nods slightly, but Dirac sniffs dismissively. "Politics has nothing to do with science. It is as useless as religious. Certainly politics here will not affect my research in England. I don't tell the German chancellor how to run his country, and he doesn't tell me how to do mathematics."

Pauli raises his eyebrows. "Spoken like a true theorist. Well, it may interest you to know that the economic depression in America has had a great effect on German politics. In last month's elections, the extremist Socialist Workers' Party won over 18% of the seats in Parliament, which is *six times* their numbers from last year. I'm afraid in these desperate times, unemployed workers are looking to bold new leadership. Bold, but dangerous, like this anti-Semite Adolf Hitler."

Dirac ceases to pay any attention to Pauli and eagers reads through Von Neumann's letter, mouthing the words silently and occasionally pausing, his mouth opening into a little smile when he understands something new that pleases him.

"Well, Dirac is happy as long as there are still jobs in quantum physics." Pauli sees Elina's notebook of equations. "Oh, fraulein, I did not expect that you were also a mathematician. I see you've been using this abominable bra-ket notation."

[2]Niels Bohr was from Denmark and to many in the physics community, the most famous Dane of his time.

4.2 Pauli's Monologue

4.2.1 A Single-Qubit Gate

Oh, but Dirac hasn't given you the whole picture. You don't even know what matrices are valid gates, or operations on a single qubit. Surely you didn't think that *any* old 2×2 matrix could be a single-qubit quantum gate?

For example, give me a single-qubit gate which takes the state $|0\rangle$, the north pole on the Bloch sphere, to the point on the equator which is $|+\rangle$.

Elina, sensing a trap, writes down the matrix and equation below.

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \tag{4.1}$$

$$\tilde{H}\,|0\rangle = |+\rangle \tag{4.2}$$

Wrong! *Both Elina and Ehrenfest wince.* See what that matrix does to the state $|1\rangle$?

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}|1\rangle = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.3}$$

That's not even a quantum state! Furthermore, what happens if you apply it twice to $|0\rangle$? *At this point, Elina is afraid to find out, but she works out the equation anyway*.

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}|0\rangle = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}|+\rangle = \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix} \tag{4.4}$$

Isn't that strange? And if keep applying this gate to the state repeatedly, it will go down by a fraction of $\frac{1}{\sqrt{2}}$ every time. Eventually it will have components of $\frac{1}{4}$, then $\frac{1}{8}$, and then vanish away altogther! So then we are back to the problem that this matrix is a quantum state killer.

4.2.2 Unitary Matrices

No, my dear, quantum gates must have the property that they take unit vectors to unit vectors. Otherwise, what's the point of requiring quantum states be normalized, if the next quantum gate could just obliterate it? This property we will call having *unit determinant*.

The *determinant* of a matrix is intuitively a measure of how much is scales (stretches or shrinks) a vector. For a 2×2 matrix M, it is simply calculated by this formula:

$$\det(M) = \left|\begin{matrix} a & b \\ c & d \end{matrix}\right| = ad - bc \tag{4.5}$$

Having unit determinant just means having a determinant with magnitude one. You may heard of these matrices I discovered to describe an extra, purely quantum degree of freedom called *spin* for certain particles like the electron. They are quite famous, people named them after me.

Dirac here rolls his eyes and mutters that he discovered them independently, but Pauli ignores him.

Well, you can verify that each of my matrices has this property of unit determinant.

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.6}$$

This is also why matrices with this property are called *unitary*.

4.2.3 Pauli Matrices

Oh, one more great property about my matrices is that together with the identity matrix they form a complete basis for all 2 unitary matrices, sometimes written as the group $U(2)$. Isn't that fascinating? That fact may come in handy for you some day.

$$U = \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \delta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1 \tag{4.7}$$

4.2.4 Adjoint Matrices as Time-Reversed Quantum Gates

Single-qubit gates have an equivalent property that you can *reverse* them. You can take my word for it that quantum mechanics is inherently time-reversible. Anything that you can do in the forward direction, you can also do in the backward direction. Well, there is one exception, but I will let Bohr tell you about that that himself tomorrow.

In other words, given a qubit (a vector on the Bloch sphere) and a single-qubit gate (a rotation on the Bloch sphere), you are able to tell two things, one about the future and one about the past.

If you *are about to apply* the gate to the qubit, then you know what the qubit will be afterwards. If you *have already applied* the gate to the qubit, then you know what the qubit was before you applied the qubit.

Furthermore, the time-reversed version of a single-qubit gate is simply the adjoint, which I see Dirac has already taught you about. Just like the adjoint of a column vector (a ket) is just its conjugate transpose, which is a row vector with all the complex entries conjugated (a bra), we can define the adjoint of a matrix as its conjugate transpose. We write the adjoint of a matrix U as $U^\dagger$ and call it "U dagger."

4.2.5 Interlude

Throughout this monologue, Dirac has not been paying attention to Pauli's monologue since he assumes that he already understands it all. Finally, he speaks up. "Herr Von Neumann has introduced this tensor product to describe large quantum systems. Have you heard about this, Pauli?"

Pauli is about to respond, with his cigar in mid-air, as his ear catches a faint trace of music, and the fat man begins to hum along to it. Dirac sighs and knows what will happen next, and that his answer will be delayed; he goes back to reading the letter.

Elina repeats, "Yes, Herr Pauli, what about the tensor product? How do we make bigger quantum gates?" She and Ehrenfest watch with confusion.

"Wait!" Pauli says dramatically, and goes over to the barista, who has just placed a plastic disc record on a Victrola phonograph, one of the new talking machines that has come across the Atlantic. "What is that? What are you playing?"

The young man looks back at the record. "That, sir? It's the new Louis Armstrong song." He scratches his head for a moment, trying to remember. "Sorry, I don't know the name. It's something strange, you know how American jazz music is."

"I know the name," Pauli says excitedly. " 'Is you is or is you ain't my baby?' and it's a fantastic song! But this is a cover. Do you happen to have the original by Louis Jordan?"

"No, sir, sorry sir." The barista is apologetic. "There's a customer, I have to go."

Pauli almost doesn't notice and is snapping his finger to the beat, rhythmically tapping his way back to the table where his companions are sitting. "I was at a dance last night, in Stuttgart," Pauli explains as he sits again, taking a puff from his cigar. "The band was playing this song, I loved it!"

"Can we please return to mathematics?" Dirac asks pointedly. "I don't understand the point of all this music, especially having multiple versions of the same song. There should just be one, best version."

"Dear Dirac, I'm not the least bit surprised you think that." Pauli lifts his wineglass to take a swig, angling his cigar away carefully to avoid ashing into his drink. "It is like having multiple ways to prove one theorem. There is beauty in multiple."

"That is not a valid analogy. In mathematics, there is always one, best proof. Sometimes it hasn't been discovered yet." But Dirac doesn't feel like pressing the point. "Besides, you haven't answered Miss Gamow's question.

Pauli nods amiably. "Yes, yes, of course. Wait, what? What tensor product?" He confers with Dirac over the letter for awhile. "Oh yes, I see what he has done. What a typical mathematician."

4.2.6 The Tensor Product for Matrices

Just like Von Neumann has taught you to combine smaller quantum states into larger quantum states using the tensor product, you can also combine gates on these smaller states into gates on the larger states.

Let's start with the simplest example: two single-qubit gates, U_1 operating on the qubit $|\psi\rangle$ and U_2 operating on the qubit $|\phi\rangle$.

$$U_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad |\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{4.8}$$

$$U_2 = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \qquad |\phi\rangle = \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \tag{4.9}$$

Now let's try to combine the two single-qubit operators into a single two-qubit operator. The tensor product between a 2×2 square matrix and another 2×2 square matrix is now a 4×4 square matrix. In general, if one matrix is $d_1 \times d_1$ and another matrix is $d_2 \times d_2$, their tensor product is a matrix which is $d_1 d_2 \times d_1 d_2$.

$$U_1 \otimes U_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{bmatrix} \tag{4.10}$$

One way to think about the tensor product of matrices is as a "matrix of matrices." The upper-left block of four elements is the 2×2 matrix U_2 scaled by the upper-left element of U_1, namely a. The upper-right block of four elements is the 2×2 matrix U_2 scaled by the upper-right element of U_1, b, and so forth.

$$\begin{bmatrix} \begin{bmatrix} ae & af \\ ag & ah \end{bmatrix} & \begin{bmatrix} be & bf \\ bg & bh \end{bmatrix} \\ \begin{bmatrix} ce & cf \\ cg & ch \end{bmatrix} & \begin{bmatrix} de & df \\ dg & dh \end{bmatrix} \end{bmatrix} \tag{4.11}$$

You can now verify that the two-qubit 4×4 matrix $U_1 \otimes U_2$ applied to the two-qubit state $|\psi\rangle \otimes |\phi\rangle$ gives the same results as operating on the separate qubits and then taking the tensor product of the resulting states.

$$(U_1 \otimes U_2)(|\psi\rangle \otimes |\phi\rangle) = (U_1 |\psi\rangle) \otimes (U_2 |\phi\rangle) \tag{4.12}$$

4.3 Problem Set

1. Verify that the Pauli matrices X, Y, and Z have unit determinant.

2. This is an improved version of Problem 6 from Chapter 2: What is the 2×2 unitary matrix that rotates the state $|0\rangle$ to the state $|+\rangle$ on the Bloch sphere? What happens if you apply this matrix twice to $|0\rangle$? What happens if you apply twice to $|1\rangle$?

3. What is the two qubit gate which is a tensor product of the Pauli X gate on the first qubit and Pauli Z on the second qubit? We write this operator as $X \otimes Z$. Is this the same as $Z \otimes X$, and if not, how are they different?

4. What is the matrix for the time-reversed operations corresponding to the matrices in Chapter 2, Problem 5? What is their geometric interpretation?

5. One of the most important operations in quantum computing is the controlled-NOT (CNOT) gate. CNOT acts as follows on the computational basis (Z-basis) states. The first ket is called the *control* qubit and the second key is called the *target* qubit. $|x\rangle$ represents either $|0\rangle$ or $|1\rangle$, which is a shorthand so we don't have to write out the same equation twice. $|\overline{x}\rangle$ means the opposite of $|x\rangle$ in the same equation, or $|1-x\rangle$.

$$CNOT\,|0\rangle\,|x\rangle = |0\rangle\,|x\rangle\,CNOT\,|1\rangle\,|x\rangle = |1\rangle\,|\overline{x}\rangle \tag{4.13}$$

We can write out its truth table on classical states as follows:

$$CNOT\,|0\rangle\,|0\rangle = |0\rangle\,|0\rangle \tag{4.14}$$
$$CNOT\,|0\rangle\,|1\rangle = |0\rangle\,|1\rangle \tag{4.15}$$
$$CNOT\,|1\rangle\,|0\rangle = |1\rangle\,|1\rangle \tag{4.16}$$
$$CNOT\,|1\rangle\,|1\rangle = |1\rangle\,|0\rangle \tag{4.17}$$

Because it operates on two qubits, which combine via the tensor product, it must be a 4×4 matrix. The CNOT gate is the same as the classical exclusive OR (XOR) in that it computes the logical XOR of the first qubit (called the *control*) and the second qubit (called the *target*) into the second qubit.

$$CNOT\,|x\rangle\,|y\rangle = |x\rangle\,|y \oplus x\rangle \tag{4.18}$$

4.4 Version History

- **16 October 2012** Original version.
- **27 November 2012** Fixed definition of unit determinant to have determinent with magnitude of one, instead of just a determinant of exactly one.