Quantum Computing for Beginners

Paul Pham with illustrations by Stephen Rice

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0.1 Instructions

Reading these entire notes and doing the problem set is meant to take at most two hours of your time before class. You don't need to read the Historical Notes, they are provided if you are curious for more details. Keep careful track of how long you spend on this assignment. If it takes you more than two hours, feel free to stop and come to class prepared to ask questions and finish the assignment with your classmates.

Chapter 5

In which we measure Heisenberg and Meitner

5.1 Göttingen Hauptbahnhof (Central Train Station)

Pauli finishes his monologue with a flourish and takes a puff from his cigar. The rays of the afternoon sun begin to cross their table as Elina asks for further details of the tensor product on unitary operators.

At last, Dirac looks at his pocket watch and primly tucks it back into his waistcoat pocket. He announces, "Heisenberg is quite late. I propose we go to our rooms and try to meet up with him later for dinner."

"Or perhaps it is our continuing to wait for him which further delays him," Pauli says with a wink.

"Then by all means, let us not keep Herr Heisenberg any longer by trying to measure his arrival," Ehrenfest jokes. They gather up their coats and papers to leave. Just as they exit the cafe, a tall, lean, athletic looking man with a short crew haircut crosses their path and bumps into Elina.

"I beg your pardon," he says, his eyes lighting up as he gives Elina the once over. "Terribly sorry, fraulein, I was in a hurry to meet—"

"Us, perhaps?" Pauli says with a laugh. "Good of you to finally notice us, Heisenberg. The Pope[1] would definitely notice if his favorite archbishop were missing from mass."

"Not religion again," sighed Dirac.

"Oh, there you all are. Pauli as loud and drunk as ever. As if *you* aren't actually the old man's favorite. And you're looking well after our Japanese adventures, Dirac.[2] Don't worry, I've tired of the religion jokes, and I've found

[1]Niels Bohr was often jokingly referred to as the spiritual head of the new quantum theory, especially the Copenhagen interpretation which many scientists believed with a religious fervor. Bohr was especially close with Heisenberg.

[2]Heisenberg and Dirac both took a world tour through Japan in 1929.

a better way to describe us. Do you like films, fraulein?" he asks Elina, ignoring Ehrenfest.

"Films?" Elina replies, who has never had the leisure to see a film back in her hometown while growing up.

"You know, motion pictures. They are fantastic!" Heisenberg becomes very animated with his hand gestures. "I have just seen the first color film in Munich, projected onto a screen stretching from wall to wall, ceiling to floor. It's from America, they are inventing the most delightful technology there even in these difficult times. The film is called *The Wizard of Oz*, and a group of friends is traveling together through a strange land to meet a great wizard. Each one of them has a wish, which is their deepest heart's desire. And just like us, there are three men, led by a brave young girl."

"Plus Ehrenfest," Pauli interjects, indicating the fifth member of their group.

"Oh, but that doesn't quite work, does it?" Heisenberg says, and immediately ignores Ehrenfest again. "I figured it out on the train ride which characters we all were, but I couldn't figure out the girl. And now here you are, Miss, er, Miss. . . "

"My name is Elina Gamow," she mumbles quietly with a slight bow.

"Elina, a pleasure," Heisenberg shakes her hand warmly. "So in a sense, we were destined to meet."

Dirac rolls his yes. "If you're quite finished, Heisenberg, I'd like to return to my room. I'm tired."

"Of course. I'll tell you more about the film over dinner tonight."

"And the elections," Pauli inserts.

"Don't get me started on the subject!" Heisenberg exclaims angrily, and then launches in on a tirade. "When I think about the direction this country is headed in. The Workers Party[3] is simply using this massive unemployment to their advantage. . . "

As the three friends stride ahead, Ehrenfest falls back where he notices Elina has become very quiet.

"Is everything all right, Miss Gamow?" he asks gently. "That Heisenberg and his reputation with women precedes him, I'm afraid. Let me know if he continues to bother you."

"He is not so bad," Elina says after awhile. "But everyone here is so brilliant, and, and. . . " she pauses, looking about suddenly, causing Ehrenfest to lean forward. She lowers her voice to say the next word, ". . . male."

"Oh. Oh I see," Ehrenfest straightens up seriously. "Well, I can certainly understand being intimidated, I'm slightly terrified myself of those three geniuses ahead of us. But you are the most promising student I've met in a long time, for one so young. And being young is an advantage in this field, let me tell you. Dirac and Heisenberg were barely 20 years old when they did their greatest work. I strongly encourage you to continue pursuing your interest in

[3]The National Socialist Workers Party, or NDSAP in German, is the official name of the Nazi party.

quantum physics, especially in combining it with computation. I don't know of anyone else doing this work."

Elina still has her doubts. "But Herr Ehrenfest, aren't there any women scientists? At least the other keypunch operators are women my age, and we can talk to each other. You've been very kind, more than kind! But even you couldn't really understand."

After a moment of thought, Ehrenfest comes to a decision. "Tomorrow, we will pay a visit to someone who I think will understand."

5.2 The Laboratory of Lise Meitner

The next day, at a few minutes until eight 'o clock, Elina stands waiting expectantly at the door to the great laboratories of Göttingen. When the clock tower strikes eight with a loud gong, the door to the laboratory opens. Behind it is Ehrenfest, who beckons for Elina to come in.

"I should have shown you the laboratories sooner. There is more to life than mathematics, I have heard," he says jokingly. They wind their way through rows and rows of benches containing intricate glassware, arranged in a maze of funnels and tubes. An oven or two puts out intense heat from one corner of the room, and every countertop is covered with weights, scales, and arcane measurement instruments of every kind.

"Please don't touch anything," Ehrenfest cautions his guest, "but if you have any questions, we can find someone for you to ask." They draw some curious stares from the men, mostly young students in their twenties, who work at the benches, and Elina becomes aware that they must make a curious sight.

They cross through several rooms, each one larger and more elaborate than the last, until finally they reach a door on the back wall. It looks like a closet, and Elina is surprised when Ehrenfest urges her forward to open the door. Inside, the room is small and narrow, packed with shelves on both sides leading back to a small slit of a window. A long table runs down the length of the room, and at the table stands a thin figure, wearing an apron and a breathing mask. The figure is covered in soot, with hands encased in thick rubber gloves as she extracts tongs holding a very hot object from a nearby furnace.

Ehrenfest waits until the hot glowing object has been placed on the counter before announcing their arrival with a loud knock.

The figure is not startled, and turns to them, removing the mask and revealing a young woman, not much older than Elina herself, with a striking face underneath her disheleved hair.

"Lise Meitner, may I present to you, Elina Gamow, who has arrived from the Ukraine to work with Herr Professor Planck." Ehrenfest gestures to his companion, who is still surprised as seeing another young woman here in the laboratory.

"A pleasure," Meitner says matter-of-factly, but not in an unfriendly way. "May I call you Elina?"

"Oh, of course," Elina says, recovering from her surprise and doing a slight curtsy. She is not sure of the protocol, if there is one, for such a rare meeting.

"Elina, you may call me Lise. I was very pleased when Ehrenfest here told me that another young woman was interested in quantum physics."

"I am also very glad to meet you. The physicists I have met have all been very kind, including Herr Ehrenfest, but I was beginning to wonder. . . " Elina trails off, not sure how to phrase her concern.

Ehrenfest takes the hint. "Well, you have a lot to discuss, I'm sure. I will see you later, Elina. A pleasure as always, Lise." He bows and exits the room.

Lise smiles. "I understand. You passed by all those great lecture halls and laboratories of Göttingen, and every scholar and scientist you see is a man. And even now, you see what preposterous conditions in which I have to scratch out a working laboratory in the physics department." She gestures around to the closet around them, but her voice is one of amused resignation and quiet determination. "Even now, I can only publish papers with a male co-author."

"Oh, that's terrible!" Elina exclaims, and Lise realizes she may have inadvertently scared the girl away from science.

"But this is important work, Elina, even if no one else recognizes it. What I'm working on now, fission of the atomic nucleus, has the potential to release great amounts of energy. Some day, it could replace coal and provide electricity for the entire world. And I started just like you, only with curiosity and a desire to make a difference."

The girl is quiet for awhile, digesting her potential future and also trying to figure out how to ask her next question. "Lise, are you. . . that is, do you have a. . . " Elina begins shyly.

"Just say it, Elina."

"Do you have a beau?" Elina blurts out. "If you don't mind me asking. You are just so pretty, and you are surrounded by so many young men." She remembers the looks of the other male scientists out in the hallway, and of Heisenberg the day before.

Meitner laughs. "I'm much too busy. There is so much work to be done. Let the men look."

5.3 Meitner's Etude

So far, you've been steeped in mathematics. Dirac and Pauli and the others are great theorists, but the true arbiter of physics is experiment. We can create many elaborate or beautiful models of Nature, but they mean nothing if they can't explain the measurements we can make. That's where experimentalists like me come in. In truth, there is no real distinction in the importance of contributions from theory or experiment, but human beings like labels. To be honest, sometimes I feel more alienated as the only experimentalist among theorists than I am at being the only women here among men.

But measurement is a tricky subject, which appears to be a the core of quantum physics. At the most basic level, measurment is just looking. I, a human scientist, look at a measuring apparatus, like weights on a scale, or a spark on the screen detector every time a photon hits it. In the classical physics of Newton, measuring a system changes nothing about the system. The subject, me, is separate from the object, and a description of its state, or its continued existence, is independent of me and my measurements. In a sense, information only flows one way, from the object to the subject.

But in reality, at the level of quanta of matter and light, such as electrons and photons, there is no such independence. Measuring a system involves having two objects bump into one another, or interact. Usually we use photons of light, which carries a certain amount of energy proportional to their frequency.[4] When we shine a lamp or a candle on an object, it is bombarded with photons. In my experiments, I'm able to create a source of light so dim that single photons can be detected one-at-a-time.

Photons are massless, but they can transmit and receive energy from interacting with other particles. Remember, Maxwell discovered that light consists of self-propagating electrical and magnetic fields, and electrons can be treated like tiny electrical charges at a point that radiate an electrical field. Therefore, a photon and an electron can interact via their electric fields, and this charge repulsion or attraction can change the trajectory of both particles.

Now, you may be wondering about this screen I have set up here behind two slits. I have been duplicating one of the greatest experiments in physics, which gets to the heart of the difference between classical and quantum descriptions of nature.

5.3.1 The Double Slit Experiment

So far, I have avoided talking about photons of light as either particles (sometimes called *corpuscles*, or "little bodies" in Latin) or waves. A particle is like a billiard ball with a definite position, momentum (a product of its mass and its directional velocity), and it interacts with other particles by bouncing against them, thus altering the trajectories of the particles involves.

A wave is like the movement of a disturbance through some medium, like waves through water. A wave has a frequency of oscillation, which tells how rapidly the disturbance shakes the medium back and forth, and an amplitude, which tells how intensely the disturbance displaces the particles of the medium at the extreme points of the oscillation (crests and troughs). A wave has no definite position in space nor mass, but it can have a direction and velocity of propagation. It can be thought of as information.

In the 1600s, Isaac Newton thought that light was made of particles. In

[4]See this thread on the Catalyst message board for a deeper explanation of photons, their frequency, and energy.

1790, Thomas Young performed a decisive experiment that showed the wave properties of light by using diffraction through two slits. This is often called "Young's double-slit experiment" and is one of the most famous scientific demonstrations of all time.

However, the quantum version of the double-slit experiment was not performed until 1909, when we perfected techniques for emitting monochromatic light (light of a single frequency, or color) with single-photon sources (light so dim, that only one photon of light is emitted at a time). What we discovered was that if the photon were allowed to scatter through the two slits, we observed the wave picture of photons with interference fringes on the detector screen. However, if we measured which slit each photon "chose" to travel through, we observed the particle picture of two sharp fringes. This is known as *wave-particle duality* and is a key tenet of Niels Bohr's so-called Copenhagen interpretation.

This destroys forever the independence of subject and object. The state of a quantum system in nature is now inextricably linked with measurement. Or to put it another way, a measurement object becomes entangled with the measurement instrument, and therefore the measurer, who is ultimately a human being. If the measurement object were in a quantum superposition of classical states, also called a *coherent* state, it would "collapse" into one of several classical states, which is the outcome that the measurer would observe. This collapse is also called *decoherence*, and is the chief source of noise in a quantum system that human beings are trying to control.

Some people have interpreted this to mean that the consciousness of a human being is what causes a quantum system to choose one classical outcome or another. However, this collapse, as far as we can tell, occurs whether or not the observer is a human being or a cat or an insect which happens to be in the way when a photon passes by. While we can characterize decoherence in fairly sophisticated ways now, why it happens at all is not very well understand.

One heuristic, or an unproven rule-of-thumb, uses the size or mass of the objects involved. If two photons interact, they are both small enough to stay in coherent quantum states. However, if a photon interacts with a human eye, or something on a similar human-sized scale, it quickly decoheres into a classical state. What is the exact relationship with the size of an object and its coherence time? Does this have anything to do with the second law of thermodynamics, the increase in entropy in the universe, and the number of disordered states available to larger objects? This is a long-standing open problem in physics, and no one knows how to completely reconcile the non-unitary, irreversible process of measurement with the rest of quantum physics, which is unitary and reversible.

Physicsts often try to unify two different phenomena as special cases of the same physical principle; in other words, to come up with the most general explanation possible, which is not always the simplest explanation. Therefore, if you ever hear of the measurement paradox, or the problem of measurement, physicists are talking about how to derive this phenomenon of measurement without making it a separate axiom of quantum mechanics.

But as for me, I don't go in for this philosophy. Bohr has one opinion of it, Einstein and others disagree with him. I just accept measurement as another rule.

5.4 Elina's Diary

That night, before she goes to bed in her dormitory room, Elina pulls out her notebook, which doubles as both her diary and her scientific journal. She opens a page that she started several weeks ago, where she has written the following:

> **Rule 1.** A quantum bit is a two-level quantum system with complex coefficients whose magnitudes squared sum to one.
>
> **Rule 2.** Two quantum systems with dimensions d_1 and d_2, respectively, combine via the tensor product to form a larger quantum system of dimension $d_1 \times d_2$.
>
> **Rule 3.** Quantum operations are reversible, unitary transformations.

Nadia, in the bunk above hers, pokes her head down. "Hey everybody, El'ka's writing secrets! Is it about a boy?" Her other roommates giggle.

Elina laughs. "No, it is not about boys. It is about the future."

"How sad, there are no boys in Elina's future," Nadia taunts her.

"There are many *men* in my future," Elina assures her.

"Many? Don't be selfish, Elishka, you should share."

"You are welcome to any of them, but some are married." She waits for the scandalized shrieks and laughter to pass. "And hopefully there will be a machine. A big, beautiful machine." The other girls don't know what to say to that.

On the page beneath the first three rules, she writes the following.

> **Rule 4.** Measurement is an irreversible projection of a quantum statement onto one of several orthonormal basis states. The probability of the outcome being a particular basis state is the magnitude of its coefficient squared.

Underneath all of it, the last thing on the page she writes is:

> I can help create the future.

5.5 Problem Set

1. Describe in your own words the double-slit experiment and how it relates to quantum physics. What is the main difference between the classical double-slit experiment and the quantum double-slit experiment?

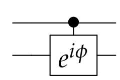
2. Draw a quantum circuit that starts with two qubits in the state $|00\rangle = |0\rangle \otimes |0\rangle$ and ends up with the entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

3. Prove that the state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is entangled by showing that it cannot be the tensor product of any three qubits $|\psi\rangle \otimes |\phi\rangle \otimes |\eta\rangle$. Hint: create symbols for the elements of the column vectors of any three qubits, try to calculate some of their values to match the state above, and find a contradiction in trying to calculate the remaining values.

4. Find the two-qubit quantum gate (4×4 unitary matrix) which performs a controlled rotation $e^{i\phi}$, which we will denote as $\Lambda(e^{i\phi})$. That is, based on the "oneness" of the control qubit, it multiplies a complex phase $e^{i\phi}$ to the "oneness" of the target qubit. That is, if we define two qubits as follows, the action of this gate is to produce an entangled two-qubit state which leaves the components for the basis states $|00\rangle$, $|01\rangle$, and $|10\rangle$ unchanged but multiplies a complex phase $e^{i\phi}$ to the component for the basis state $|11\rangle$.

$$
\begin{aligned}
|\psi\rangle &= \alpha\,|0\rangle + \beta\,|1\rangle \\
|\phi\rangle &= \gamma\,|0\rangle + \delta\,|1\rangle \\
\Lambda(e^{i\phi})\,|\psi\rangle \otimes |\phi\rangle &= \alpha'\,|00\rangle + \beta'\,|01\rangle + \gamma'\,|10\rangle + e^{i\phi}\delta'\,|11\rangle
\end{aligned}
$$

5. We can draw the quantum circuit for this gate as follows, with the first qubit being the control and the second qubit being the target.



What are the similarities and differences between this gate and the CNOT gate that you found in the previous chapter? Use this information to construct a two-qubit controlled-U gate, where U is the following single-qubit gate:

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{5.1}$$

The controlled-U gate acts by leaving the target qubit unchanged based on the "zeroness" of the control qubit, and enacting U on the target qubit based on the "oneness" of the control qubit. All gates of this form are entangling, in that they produce an entangled qubit state as a result. The ability to create this entanglement in a particular way is a key part of the power of quantum computing.

6. The following states can be called the quantum Fourier transform (QFT) states on 2 qubits.

$$\begin{aligned}
|\Phi_0\rangle &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
|\Phi_1\rangle &= \frac{1}{2}(e^{i0}|00\rangle + e^{i\pi/2}|01\rangle + e^{i\pi}|10\rangle + e^{i3\pi/2}|11\rangle \\
|\Phi_2\rangle &= \frac{1}{2}(e^{i0}|00\rangle + e^{i\pi}|01\rangle + e^{i0}|10\rangle + e^{i\pi}|11\rangle \\
|\Phi_3\rangle &= \frac{1}{2}(e^{i0}|00\rangle + e^{i3\pi/2}|01\rangle + e^{i\pi}|10\rangle + e^{i\pi/2}|11\rangle
\end{aligned}$$

These represent an alternative 2-qubit basis. Find the unitary matrix that changes basis from the QFT states on 2 qubits to the standard basis. That is, find the 4×4 unitary matrix M which acts as follows:

$$\begin{aligned}
M|00\rangle &= |\Phi_0\rangle \\
M|01\rangle &= |\Phi_1\rangle \\
M|10\rangle &= |\Phi_2\rangle \\
M|11\rangle &= |\Phi_3\rangle
\end{aligned}$$

This matrix is known as the QFT operation on 2 qubits.

7. Generalize the matrix for the QFT in the previous problem to three qubits, and then n qubits. Use the symbol ω for the (2^n)th root of unity:

$$\omega = e^{2\pi i/2^n} \tag{5.2}$$

That is, give an 8×8 matrix which is the QFT on 3 qubits, and describe what the $2^n \times 2^n$ matrix would look like for a QFT on n qubits.

8. **Optional bonus problem** If you have seen or read *The Wizard of Oz*, identify the characters of Dirac, Pauli, and Heisenberg with those of the Scarecrow, the Tin Man, and the Cowardly Lion. Justify your answer. (There is no one correct answer).

9. **Optional bonus problem** A 1-qubit state has two (real) degrees of freedom, as discussed in Chapter 2, and can be represented as the surface of

a 3-dimensional unit sphere (the set of all points in 3-dimensional space at distance one from an origin). How many real degrees of freedom does an n-qubit state have, and what is the dimensionality of the unit sphere whose surface represents all such states?

5.6 Version History

- **6 November 2012** Original version.
- **27 November 2012** Fixed QFT basis states in Problem 6.