Quantum Computing for Beginners

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0.1 Instructions

This chapter, and all of its problems, are *optional*. You should read and do the problems if you who wish to understand quantum circuits better, for example, to do problems in Chapter 5 or in later chapters.

Reading these entire notes and doing the problem set is meant to take at most two hours of your time before class. You don't need to read the Historical Notes, they are provided if you are curious for more details. Keep careful track of how long you spend on this assignment. If it takes you more than two hours, feel free to stop and come to class prepared to ask questions and finish the assignment with your classmates.

Chapter 6

In which we review quantum circuits

6.1 A Letter to Von Neumann

Dear Johnny,

These matrices keep getting bigger and bigger, if we want to deal with any decent number of qubits. If we are to build a quantum computer, we'll need a more compact way to represent quantum gates, the order in which they are executed, and which qubits they operate on. Therefore, I'm proposing the following *quantum circuit* diagrams.

6.1.1 Single-Qubit Circuits

Each qubit is represented by one horizontal line, and time is represented going from left to right. The qubits start out in an initial state, usually $|0\rangle$ on the left hand side, and gates are represented by rectangles, which cover up the lines corresponding to the qubits they apply to. So it's a little like sheet music.

The simplest circuit is a single qubit, where nothing happens to it. So it starts out as $|0\rangle$ and ends up as $|0\rangle$.

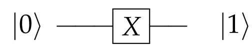
$$|0\rangle \;\text{———}\; |0\rangle \tag{6.1}$$

The line is just equivalent to a *quantum wire*, in that it carries a qubit's state forward in time. You can apply a single-qubit gate to a qubit by drawing a square which covers up just that qubit. That square represents a 2×2 unitary matrix. We can redraw the circuit above with an identity gate I, and it would still be the same circuit.

$$|0\rangle \;\text{—}\boxed{I}\text{—}\; |0\rangle \tag{6.2}$$

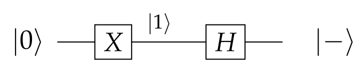
Observation 1. *Every single-qubit circuit has associated with it a* 2×2 *unitary matrix.*

Here's a circuit where the Pauli X gate is applied to our qubit, changing $|0\rangle$ to $|1\rangle$.



(6.3)

You can even apply multiple gates, one after the other, to a qubit, by drawing the squares one after the other. Squares to the left occur first in time, and squares to the right occur later. Here is a circuit which extends the previous circuit by applying a Hadamard to the qubit. I label the intermediate state $|1\rangle$ in between the X and H gates to show you what that qubit's state is at that point in time.



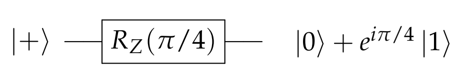
(6.4)

You can determine the matrix for this circuit simply by multiplying the matrices from left to right. In this case, the matrix for this circuit is:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{6.5}$$

Observation 2. *Now here's something strange. We apply gates to single qubits by multiplying a matrix on the left to a column vector on the right. However, in the circuit, we start with an input state on the left, and apply gates to it going to the right, ending in a final state. This is an unfortunate inconsistency, but in the analogy to sheet music, most people read diagrams from left-to-right. Therefore, please forgive me, and keep in mind that the direction of matrix-vector multiplication is opposite from the way it appears in the circuit. If you like, you can write the input qubits on the right and the output qubits on the left, and have time flow from right-to-left, but that way feels less intuitive to me.*

Since we start out in the classical state $|0\rangle$, and most gates are controlled on the $|1\rangle$ component of a qubit, it's useful to first create an equal superposition of $|0\rangle$ and $|1\rangle$, and then do something to the $|1\rangle$ component. For example, assuming we have a $|+\rangle$ state, here is a circuit which rotates the $|1\rangle$ component by a phase of $e^{i\pi/4}$:



(6.6)

where the square labeled $R_Z(\pi/4)$ represents this matrix:

$$R_Z(\pi/4) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \tag{6.7}$$

6.1.2 Two-Qubit Circuits

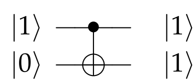
Okay, simple enough. But now let's add a second qubit, and see how we would draw a two-qubit gate. As a sanity check, let's see what the simplest two-qubit circuit would look like.

$$\begin{array}{ccc} |0\rangle & \text{———} & |0\rangle \\ |0\rangle & \text{———} & |0\rangle \end{array} \tag{6.8}$$

This circuit just corresponds to a 4×4 identity matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.9}$$

Now what about the quantum circuit for the important CNOT gate?



(6.10)

In the circuit above, we start with the state $|1\rangle \otimes |0\rangle$, and we end with the state $|1\rangle \otimes |1\rangle$. The top qubit is called the *control*, which is what the dot signifies. Based on the $|1\rangle$ component of the control qubit, the bottom qubit, called the *target*, will be flipped, or have the Pauli (X) gate applied to it.

The above states are purely classical states though, so the input and output two-qubit states are separable into the tensor product of single-qubit states. If we put in a general single-qubit state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ as the control and $|\phi\rangle = \gamma |0\rangle + \delta |1\rangle$ as the target. Then in general, the CNOT gate is *entangling*, in that it produces an entangled state that cannot be separated as a tensor product of two single-qubit states. I won't work out the details here, but you may wish to. Ha, now I am assigning homework problems to you. Who is the student and who is the master now? (That is a rhetorical question, please don't answer.)

Problem 1. *Show that the CNOT gate, when given a tensor product of two single-qubit states as shown above, produces an entangled two-qubit state as output. In what special cases does the CNOT not produce an entangled state?*

But we can also consider two separate single-qubit gates acting on two separate qubits independently.

$$\begin{array}{ccc} \alpha |0\rangle + \beta |1\rangle & \text{—}\boxed{X}\text{—} & \beta |0\rangle + \alpha |1\rangle \\ \gamma |0\rangle + \delta |1\rangle & \text{—}\boxed{Z}\text{—} & \gamma |0\rangle - \delta |1\rangle \end{array} \tag{6.11}$$

In this case, the circuit above is equivalent to acting on the two qubit state $|\psi\rangle \otimes |\phi\rangle$ with the 4×4 unitary matrix $X \otimes Z$.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{6.12}$$

Observation 3. *A quantum circuit is just shorthand for a unitary matrix. If the circuit has 2 qubits, then the equivalent unitary matrix has dimension 4×4.*

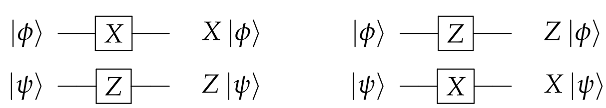
Problem 2. *If a quantum circuit has n qubits, what is the size of the equivalent unitary matrix?*

6.1.3 Order Matters

Note that it matters which qubit we regard as "first" (on the top line, or first in our tensor product) and which qubit we regard as "second" (on the bottom line, or second in our tensor product). As I learned earlier, applying a Z gate on the first matrix and an X gate on the second matrix is represented by a two-qubit unitary matrix $Z \otimes X$, which is different from Equation 6.12.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{6.13}$$

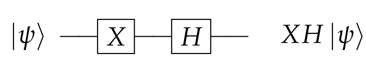
Likewise, the two circuits look similar, but are not equivalent.



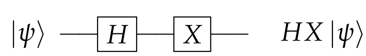
(6.14)

Problem 3. *Show that $X \otimes Y |\psi\rangle \otimes |\phi\rangle$ produces the same state as $Y \otimes X |\phi\rangle \otimes |\psi\rangle$. Draw a quantum circuit for each of these two cases. How are they related?*

Another way in which order matters is that any two gates in general do not *commute*. To see this, consider the following two, non-equivalent circuits.



(6.15)



(6.16)

To see this, consider that each circuit is equivalent to a unitary matrix, and that these matrices are not equivalent.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \neq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{6.17}$$

This is equivalent to the observation that matrix multiplication in general does not commute ($AB \neq BA$).

6.1.4 Swap Gate Revisited

Another way to see the effect of switching the first and second qubit is to examine the *SWAP* gate, which is defined as the gate which switches the first and second qubit in a tensor product.

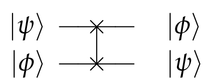
$$SWAP \left|\psi\right\rangle \otimes \left|\phi\right\rangle = \left|\phi\right\rangle \otimes \left|\psi\right\rangle \tag{6.18}$$

Remember that I derived it by calculating the 4×4 unitary matrix *SWAP* that acted as follows on $\left|\psi\right\rangle \otimes \left|\phi\right\rangle$.

$$SWAP \left|\psi\right\rangle \otimes \left|\phi\right\rangle = SWAP \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{bmatrix} = SWAP \begin{bmatrix} \gamma\alpha \\ \delta\alpha \\ \gamma\beta \\ \delta\beta \end{bmatrix} = \begin{bmatrix} \gamma\alpha \\ \gamma\beta \\ \delta\alpha \\ \delta\beta \end{bmatrix} = \left|\phi\right\rangle \otimes \left|\psi\right\rangle \tag{6.19}$$

The matrix which accomplishes this is given below, and is drawn in a quantum circuit as two $\times$ symbols connected by a vertical line.

$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.20}$$



(6.21)

Now that we have gotten the hang of the order of qubits in the tensor product and how they appear in a quantum circuit, we can just neglect labeling the first and second qubit.

Observation 4. *The qubit labels in a quantum circuit are arbitrary, but it matters which gate is applied to which qubit. That is, if we change the qubits that each gate operates on, or even the order of the gates, we create a different, non-equivalent unitary matrix (in general).*

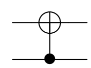
After some study, I noticed an interesting relationship between *SWAP* and *CNOT*. First let's describe the action of *CNOT* with control on the first qubit and target on the second qubit, which we can write more specifically as $CNOT(1,2)$.

$$CNOT(1,2) \left|\psi\right\rangle \otimes \left|\phi\right\rangle = CNOT(1,2) \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{bmatrix} = \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\delta \\ \beta\gamma \end{bmatrix} \tag{6.22}$$

The $|1\rangle$ component of $|\psi\rangle$, which is the control qubit, has a coefficient of β. Therefore, for the two vector components which depend on β, we switch the one dependent on γ with the one dependent on δ. Note that at the end of Equation 6.22 we cannot write the state as the tensor product of two single-qubit states, because now it is entangled, in general.

Problem 4. *Write an equation similar to Equation 6.22 but demonstrating the action of CNOT with control on the second qubit $|\phi\rangle$ and target on the first qubit $|\psi\rangle$. (Hint: The $|1\rangle$ component of $|\phi\rangle$ is now δ, and you want to switch the components dependent on α and β.*

We can draw the circuit for this "reversed" $CNOT$, or $CNOT(2,1)$, like so:



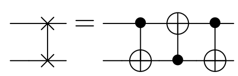
(6.23)

6.1.5 The Reversed CNOT Matrix

Is the reversed CNOT the same as the CNOT? That is, does it matter which qubit is the control and which is the target? Let's find out, by way of a few observations.

Due to the results of Problem 4 and Equation 6.19, we can decompose $SWAP$ into three $CNOT$ gates as shown below, purely by observing the effect of the gates on the column vector components.

Observation 5. *A SWAP gate is equivalent to the following three CNOT gates.*

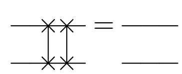


(6.24)

Therefore, the matrices for these two circuits must also be equal.

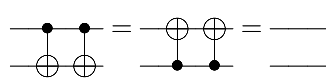
We make the following two observations.

Observation 6. *Performing two SWAP gates twice in a row cancels out to the identity.*



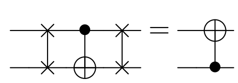
(6.25)

Observation 7. *Performing two CNOT gates, in any direction, twice in a row cancels out to the identity.*



(6.26)

Problem 5. *What is the matrix for the reversed CNOT, that is, $CNOT(2,1)$? Use the following observation:*



(6.27)

Well, that is all I have been able to think about for now. Hopefully this letter, and these quantum circuit diagrams are useful to you. I think in the future they will be a useful way of describing, or even coming up with, new quantum algorithms.

Problem 6. *What are the advantages of drawing a quantum circuit diagram versus using matrices? When would you prefer to use one and not the other?*

Warm regards,
Elina

6.2 Acknowledgements

These and all other notes use the Qcircuit LaTeX package by Bryan Eastin and Steve Flammia.